conductivity; α, heat-transfer coefficient; Q, internal source strength; φ and f, temperature distributions at beginning of process and on boundary of domain respectively; ν, outward normal to boundary Γ_2; γ, temperature gradient; q, heat flux density; η, fixed point on boundary Γ_2; $D \equiv D_1 X D_2$; $\Omega \equiv DX(0, t_{ob}]$, $\Omega_1 \equiv D_1 X(0, t_{ob}]$.

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FINITE INTEGRAL TRANSFORMS FOR HEAT AND MASS TRANSFER PROBLEMS IN NONSTATIONARY AND INHOMOGENEOUS MEDIA

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UDC 536.2

Several boundary-value problems of heat and mass transfer are solved for equations with varying coefficients.

Integral transforms are widely used in solving transport problems, mostly described by equations with constant coefficients. Important contributions in developing the method of finite integral transforms were made by Grinberg [1], Tranter [2], the authors of [3-6], etc.

In the present study we construct finite integral transforms for several boundary-value problems of heat and mass transfer, described by equations with varying coefficients. Kernels and norms of the transforms and characteristic equations for finding eigenvalues are determined for these problems. In this case it is important to develop an approach to solving these equations, as suggested by the present author [7].

Consider the problem

$$a(t) f(r) \frac{\partial T}{\partial t} = b(t) \frac{1}{r^\nu} \frac{\partial}{\partial r}\left[r^\nu \lambda(r) \frac{\partial T}{\partial r}\right] + b(t) \varphi(r) \frac{\partial T}{\partial r} + g(r, t) T + W(r, t), \tag{1}$$

$$R_1 < r < R_2, \left[\lambda(r) \frac{\partial T}{\partial r} + \alpha_1 T \right]_{r=R_1} = \beta_1(t),$$
$$\left[\lambda(r) \frac{\partial T}{\partial r} + \alpha_2 T \right]_{r=R_2} = \beta_2(t), \tag{2}$$

where α_1, α_2 are constant, and $\nu = 0, 1, 2$ are shape coefficients of the geometric region. We introduce the notation

$$\int_{R_1}^{R_2} r^\nu \Phi(pr) T(r, t)\, dr = \bar{T}(p, t), \tag{3}$$

Institute of Technical Thermophysics, Academy of Sciences of the Ukrainian SSR, Kiev. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 41, No. 1, pp. 149-157, July, 1981. Original article submitted February 29, 1980.

where $\Phi(pr)$ is a kernel of a finite integral transform to be determined. Applying the scheme of [1-7], we find that $\Phi(pr)$ and $\bar{T}(p, t)$ satisfy the equations

$$\frac{d}{dr}\left[\lambda(r)\frac{d\Psi(pr)}{dr}\right]-\frac{d}{dr}[\varphi(r)\Psi(pr)]+\frac{\nu}{r}\lambda(r)\frac{d\Psi(pr)}{dr}-\frac{\nu}{r}\varphi(r)\Psi(pr)+p^2\Phi(pr)=0,\quad \Psi(pr)\equiv\frac{\Phi(pr)}{f(r)},\tag{4}$$

$$a(t)\frac{d\bar{T}(p, t)}{dt}\approx -b(t)p^2\bar{T}(p, t)+Z(t)\bar{T}(p, t)+\bar{W}(p, t)+K(p, t).\tag{5}$$

In (5) we denoted

$$Z(t)\equiv\frac{1}{R_2-R_1}\int_{R_1}^{R_2}\frac{g(r, t)}{f(r)}dr,\quad \bar{W}(p, t)\equiv\int_{R_1}^{R_2}r^{\nu}\Phi(pr)\frac{W(r, t)}{f(r)}dr;\tag{6}$$

$$K(p, t)=b(t)\left[r^{\nu}\lambda(r)\Psi(pr)\frac{\partial T(r, t)}{\partial r}-r^{\nu}\lambda(r)T(r, t)\frac{d\Psi(pr)}{dr}+r^{\nu}\varphi(r)\Psi(pr)T(r, t)\right]_{R_1}^{R_2}.\tag{7}$$

From the independence condition of $K(p, t)$ on values of the unknown function $T(r, t)$ with account of boundary conditions (2), we obtain that $\Psi(pr)$ must satisfy the relations

$$\lambda(R_1)\left.\frac{d\Psi(pr)}{dr}\right|_{r=R_1}+[\alpha_1-\varphi(R_1)]\Psi(pR_1)=0,\tag{8}$$

$$\lambda(R_2)\left.\frac{d\Psi(pr)}{dr}\right|_{r=R_2}+[\alpha_2-\varphi(R_2)]\Psi(pR_2)=0.\tag{9}$$

In this case $K(p, t)$ acquires the form

$$K(p, t)=b(t)[R_2^{\nu}\Psi(pR_2)\beta_2(t)-R_1^{\nu}\Psi(pR_1)\beta_1(t)].\tag{10}$$

Equation (4) reduces to the following:

$$\lambda(r)\frac{d^2\Psi}{dr^2}+\left(-\varphi+\frac{\nu}{r}\lambda+\frac{d\lambda}{dr}\right)\frac{d\Psi}{dr}+\left(p^2f-\frac{\nu}{r}\varphi-\frac{d\varphi}{dr}\right)\Psi=0.\tag{11}$$

Consider first the case

$$\lambda(r)=\lambda_0r^{n-\varepsilon},\quad f(r)=r^{m-\delta},\quad \varphi(r)=\varphi_0r^{n-\varepsilon-1},\quad 0\leqslant\varepsilon,\ \delta\leqslant 1.\tag{12}$$

Here λ_0, φ_0 are constants, and m and n are arbitrary integers. Taking into account (12), Eq. (11) has a solution

$$\Psi(pr)=Ar^{\frac{1-a}{2}}Z_s\left(\frac{2}{M}\sqrt{b}\,r^{\frac{M}{2}}\right)+Br^{\frac{1-a}{2}}Z_{-s}\left(\frac{2}{M}\sqrt{b}\,r^{\frac{M}{2}}\right).\tag{13}$$

Here A and B are constants to be determined, and Z_S is a cylindrical function whose analytic definition depends on the constants appearing in the equation:

$$a\equiv n-\varepsilon+\nu-\frac{\varphi_0}{\lambda_0},\quad s\equiv\frac{1}{M}[(1-a)^2-4c]^{1/2},\tag{14}$$

$$M\equiv m-\delta-(n-\varepsilon)+2,$$

$$c\equiv-\frac{\varphi_0}{\lambda_0}(n-\varepsilon+\nu-1),\ b\equiv\frac{p^2}{\lambda_0},\tag{15}$$

some of which can be either real or complex. It follows from (8) that

$$\frac{B}{A} = -\left\{[\alpha_1 - \varphi(R_1)]\, R_1^{\frac{1-a}{2}} Z_s(y)|_{r=R_1} + \lambda(R_1) \left[\frac{1-a}{2} R_1^{\frac{1-a-2}{2}} Z_s(y)|_{r=R_1} + R_1^{\frac{1-a}{2}} \frac{dZ_s(y)}{dr}\Big|_{r=R_1}\right]\right\} \times$$

$$\times \left\{[\alpha_1 - \varphi(R_1)]\, R_1^{\frac{1-a}{2}} Z_{-s}(y)|_{r=R_1} + \lambda(R_1) \left[\frac{1-a}{2} R_1^{\frac{1-a-2}{2}} Z_{-s}(y)|_{r=R_1} + R_1^{\frac{1-a}{2}} \frac{dZ_{-s}(y)}{dr}\Big|_{r=R_1}\right]\right\}^{-1} \equiv F(R_1), \tag{16}$$

where $y \equiv \frac{2}{M} \frac{P_k}{\sqrt{\lambda_0}} r^{\frac{M}{2}}$. In what follows we use as transform kernel the expression

$$\Phi(p_k r) = r^{\frac{1-a}{2}} \left[Z_s(y) + \frac{B}{A} Z_{-s}(y)\right] f(r), \tag{17}$$

in which B/A is determined by relationship (16). The eigenvalues p_k are the positive roots of the transcendental equation

$$F(R_1) = -\left\{[\alpha_2 - \varphi(R_2)]\, R_2^{\frac{1-a}{2}} Z_s(y)|_{r=R_2} + \lambda(R_2) \left[\frac{1-a}{2} R_2^{\frac{1-a-2}{2}} Z_s(y)|_{r=R_2} + R_2^{\frac{1-a}{2}} \frac{dZ_s(y)}{dr}\Big|_{r=R_2}\right]\right\} \times$$

$$\times \left\{[\alpha_2 - \varphi(R_2)]\, R_2^{\frac{1-a}{2}} Z_{-s}(y)|_{r=R_2} + \lambda(R_2) \left[\frac{1-a}{2} R_2^{\frac{1-a-2}{2}} Z_{-s}(y)|_{r=R_2} + R_2^{\frac{1-a}{2}} \frac{dZ_{-s}(y)}{dr}\Big|_{r=R_2}\right]\right\}^{-1}, \tag{18}$$

obtained from condition (9). Here function $F(R_1)$ is determined by expression (16).

The norm $N^2(p_k)$ of the transform is

$$N^2(p_k) \equiv \int_{R_1}^{R_2} r^\nu \Phi^2(p_k r)\, dr = -\frac{1}{p_k^2}\left[r^\nu \lambda \Phi \frac{d\Psi}{dr} - r^\nu \varphi \Phi \Psi\right]_{R_1}^{R_2} + \frac{1}{p_k^2} \int_{R_1}^{R_2} \left(r^\nu \lambda \frac{d\Psi}{dr}\frac{d\Phi}{dr} - r^\nu \varphi \Psi \frac{d\Phi}{dr}\right) dr. \tag{19}$$

The exact expression for the integral in (19) is quite awkward. Using the mean value theorem, we have approximately

$$N^2(p_k) \approx \frac{1}{p_k^2}\left\{\left[\alpha_2 + \frac{\lambda_0}{\gamma} \frac{R_2^{\gamma-\nu}}{(R_2-R_1)^2} - \frac{R_2^{-\nu}}{R_2-R_1} \frac{\Omega(p_k, R_1, R_2)}{\Psi(p_k R_2)}\right] \times\right.$$

$$\left.\times R_2^\nu \Phi(p_k R_2)\Psi(p_k R_2) - \left[\alpha_1 + \frac{\lambda_0}{\gamma} \frac{R_1^{\gamma-\nu}}{(R_2-R_1)^2} - \frac{R_1^{-\nu}}{R_2-R_1} \frac{\Omega(p_k, R_1, R_2)}{\Psi(p_k R_1)}\right] R_1^\nu \Phi(p_k R_1)\Psi(p_k R_1)\right\}, \tag{20}$$

$$\gamma \equiv \nu + n - \varepsilon + 1, \quad \Omega(p_k, R_1, R_2) = \int_{R_1}^{R_2} r^\nu \frac{\varphi(r)}{f(r)} \Phi(p_k r)\, dr. \tag{21}$$

The solution of Eq. (5) is

$$T(p_k, t) = \left[T(p_k, 0) + \int_0^t E(p_k, t') \exp\left(\int_0^{t'} n(p_k, t'')\, dt''\right) dt'\right] \exp\left(-\int_0^t n(p_k, t')\, dt'\right), \tag{22}$$

$$E(p_k, t) = \frac{1}{a(t)}\left[\bar{W}(p_k, t) + K(p_k, t)\right], \tag{23}$$

$$\bar{T}(p_k, 0) = \int_{R_1}^{R_2} r^\nu T_0(r)\, \Phi(p_k r)\, dr,$$

$$n(p_k, t) = \frac{b(t)}{a(t)} p_k^2 - \frac{Z(t)}{a(t)}, \tag{24}$$

therefore the required function is

$$T(r,\ t)=\sum_{k=1}^{\infty}\bar{T}(p_k,\ t)\,\Phi(p_k r)\,N^{-2}(p_k). \tag{25}$$

The summation here is performed over the eigenvalue p_k subscript, being the positive increasing roots of Eq. (18).

Consider the more general equation

$$a(t)\,f(r)\,\frac{\partial T}{\partial t}=b(t)\,\frac{h(r)}{r^{\nu}}\,\frac{\partial}{\partial r}\left[r^{\nu}\lambda(r)\,\frac{\partial T}{\partial r}\right]+b(t)\,\varphi(r)\,\frac{\partial T}{\partial r}+g(r,\ t)\,T+W(r,\ t). \tag{26}$$

As previously, the transformant of the required function is determined by relationship (3). Applying the approach discussed above, we find that the function $\Psi(pr)\equiv\Phi(pr)/f(r)$ satisfies the equation

$$\lambda h\,\frac{d^2\Psi}{dr^2}+\left(2\lambda\,\frac{dh}{dr}+h\,\frac{d\lambda}{dr}+\frac{\nu}{r}\,\lambda h-\varphi\right)\frac{d\Psi}{dr}+$$
$$+\left(\frac{d\lambda}{dr}\,\frac{dh}{dr}+\lambda\,\frac{d^2h}{dr^2}+\frac{\nu}{r}\,\lambda\,\frac{dh}{dr}-\frac{d\varphi}{dr}-\frac{\nu}{r}\,\varphi+p^2f\right)\Psi=0. \tag{27}$$

Consider the case when the functions $\lambda(r)$, $f(r)$ are determined by relationship (12), and $h(r)$, $\varphi(r)$ are $h(r)=h_0r^{k-\mu}$, $\varphi(r)=\varphi_0r^{l-\sigma}$, where k, l are arbitrary integers, and $0\le\mu,\ \sigma\le1$. Requiring that the equality $l-\sigma-(n-\varepsilon)-(k-\mu)=-1$ be satisfied, Eq. (27) has a solution determined by relationship (13). The coefficients a, s, M, c, b, however, which were earlier determined by expressions (14) and (15), acquire for Eq. (26) the form

$$a=2(k-\mu)+n-\varepsilon+\nu-\frac{\varphi_0}{\lambda_0h_0}, \tag{28}$$
$$s=\frac{1}{M}\,[(1-a^2)-4c]^{1/2},\quad b=\frac{p_k^2}{\lambda_0h_0},$$

$$M=m-\delta-(n-\varepsilon)+2-(k-\mu),\quad c=(k-\mu)\,(n-\varepsilon+k-\mu-1+\nu)-\frac{\varphi_0}{\lambda_0h_0}\,(l-\sigma+\nu). \tag{29}$$

If the following boundary conditions are given for (26)

$$\left[\lambda(r)\,h(r)\,\frac{\partial T}{\partial r}+\alpha_1T\right]_{r=R_1}=\beta_1(t),$$
$$\left[\lambda(r)\,h(r)\,\frac{\partial T}{\partial r}+\alpha_2T\right]_{r=R_2}=\beta_2(t), \tag{30}$$

the ratio B/A and the characteristic equation for finding eigenvalues for the kernel of the transform of equation (26) are determined, with account of (28) and (29), by expressions (16) and (18), replacing in them $\lambda(r)$ by $\lambda(r)\cdot h(r)$. The norm of this transform is

$$N^2(p_k)=-\frac{1}{p_k^2}\left[r^{\nu}\lambda\Phi\,\frac{d}{dr}(h\Psi)-r^{\nu}\varphi\Phi\Psi\right]_{R_1}^{R_2}+\frac{1}{p_k^2}\int_{R_1}^{R_2}\left[r^{\nu}\lambda\,\frac{d\Phi}{dr}\,\frac{d}{dr}(h\Psi)-r^{\nu}\varphi\Psi\,\frac{d\Phi}{dr}\right]dr. \tag{31}$$

Since

$$\lambda(R_i)\,h(R_i)\,\frac{d\Psi(pr)}{dr}\bigg|_{r=R_i}+\left[\alpha_i+\lambda(R_i)\,\frac{dh(r)}{dr}\bigg|_{r=R_i}-\varphi(R_i)\right]\Psi(pR_i)=0, \tag{32}$$

where $i=1,\ 2$, then approximately

$$N^2(p_k)\approx\frac{1}{p_k^2}\left\{\left[\alpha_2+\frac{R_2^{\gamma-\nu}}{(R_2-R_1)^2}\,\frac{\lambda_0}{\gamma}\,h(R_2)\;-\frac{R_2^{-\nu}}{R_2-R_1}\,\frac{\Omega(p_k,\ R_1,\ R_2)}{\Psi(p_kR_2)}\right]R_2^{\nu}\Phi(p_kR_2)\,\Psi(p_kR_2)-\right.$$

$$-\left[\alpha_1+\frac{R_1^{\gamma-\nu}}{(R_2-R_1)^2}\,\frac{\lambda_0}{\gamma}\,h(R_1)\;-\frac{R_1^{-\nu}}{R_2-R_1}\,\frac{\Omega(p_k,\,R_1,\,R_2)}{\Psi(p_kR_1)}\right]R_1^{\nu}\Phi(p_kR_1)\,\Psi(p_kR_1)\Big\}\,. \tag{33}$$

By (26) the transformant T(r, t) has the form (22), where the functions $E(p_k, t)$, $\bar{T}(p_k, 0)$, $n(p_k, t)$ and $K(p_k, t)$ are determined by expressions (23), (24), and (10). In this case the transform kernel is chosen to be (17), where a, s, b, M, and c are determined by relations (28) and (29), and the function $\lambda(r)$ is replaced by $\lambda(r)h(r)$. The solution of Eq. (26) is of the form (25), in which case the norm of the transform is chosen by expression (33).

The transfer equations under consideration have a very wide range of application. They underlie mathematical models of turbulent transfer in the atmosphere and of many thermotechnological processes. Equations for geopotential tendency, used for weather prediction, as well as many problems of convective thermal conductivity (diffusion) reduce to them.

The method suggested for solving these equations is easily realized for the region $0 \le r \le R$ and for other combinations of boundary conditions.

A method was suggested in [7] of solving the equation of convective diffusion with a source in the form of an arbitrary function of coordinates. The method is generalized below to the case of the more general problem

$$u(x,\,y)\,\frac{\partial C}{\partial x}+v(x,\,y)\,\frac{\partial C}{\partial y}=D_0\,\frac{\partial}{\partial y}\left[\gamma(x,\,y)\,\frac{\partial C}{\partial y}\right]+f(x,\,y)\,C+F(x,\,y), \tag{34}$$

$$C|_{y=0}=0,\quad \left.\frac{\partial C}{\partial y}\right|_{y=\frac{H}{2}}=0,\quad C(0,\,y)=C_0(y). \tag{35}$$

Equation (34) describes convective diffusion in a flow in a planar channel of width H under conditions of total absorption of the diffusing component at the channel walls. The coordinate dependence of the diffusion coefficient $D = D_0\gamma(x, y)$, D_0 = const must be accounted for, e.g., when the temperature and pressure in the flow depend strongly on coordinates.

We choose a stream function Ψ, such that $u = \partial\Psi/\partial y$, $v = -\partial\Psi/\partial x$, and transform from the coordinate system (x, y) to the system (x, Ψ), in which

$$\left.\frac{\partial C}{\partial x}\right|_y=\left.\frac{\partial C}{\partial x}\right|_{\Psi}-v\,\frac{\partial C}{\partial \Psi}\,,\quad \frac{\partial C}{\partial y}=u\,\frac{\partial C}{\partial \Psi}\,, \tag{36}$$

and Eq. (34) is

$$\frac{\partial C}{\partial x}=D_0\,\frac{\partial}{\partial \Psi}\left[\gamma(x,\,\Psi)\,u\,\frac{\partial C}{\partial \Psi}\right]+\frac{1}{u}\,f(x,\,\Psi)\,C+\frac{1}{u}\,F(x,\,\Psi). \tag{37}$$

Let $u=\beta\,\frac{m(\Psi)}{n(x)}$ (the case $m(\Psi)=\Psi^{\frac{1}{2}}$, $n(x)=x^{\frac{1}{4}}$ was considered in [7]), and $\gamma(x, \Psi) = Q(\Psi)N(x)$. We apply the scheme of finite integral transforms; we find that the kernel of the transform $\Phi(p\Psi)$ and the transformant of the required function satisfy the equations

$$\frac{d}{d\Psi}\left[m(\Psi)\,Q(\Psi)\,\frac{d\Phi}{d\Psi}\right]=-p^2\Phi, \tag{38}$$

$$n(x)\,\frac{d\bar{C}}{dx}=-\beta D_0p_k^2N(x)\,\bar{C}+\frac{1}{\beta}\,n^2(x)\,Z(x)\,\bar{C}\;+\frac{1}{\beta}\,n^2(x)\,\bar{W}(x,\,p_k)-K(p_k,\,x), \tag{39}$$

$$K(p_k,\,x)=n(x)\,q(x)\,\Phi(p\Psi)|_{\Psi=0}\,,\quad q(x)=D(x)\left.\frac{\partial C}{\partial y}\right|_{y=0},$$

$$D(x)=D_0N(x)\,Q(\Psi_*),\quad D_0=D\Psi_*^{-m+\delta},\quad \Psi_*=\mathrm{const}\ll\Psi_0,$$

$$\bar{C}=\int_0^{\Psi_0}C(x,\,\Psi)\,\Phi(p_k\Psi)\,d\Psi,\qquad Z(x)=\frac{1}{\Psi_0}\int_0^{\Psi_0}\frac{f(x,\,\Psi)}{m(\Psi)}\,d\Psi;$$

$$\overline{W}(x,\ p_k)=\int_0^{\Psi_0}\frac{F(x,\ \Psi)}{m(\Psi)}\Phi d\Psi. \tag{40}$$

Since in variables (x, Ψ) conditions (35) are of the form $C|_{\Psi=0}=0$, $\partial C/\partial\Psi|_{\Psi=\Psi_0}=0$, $\Psi_0=\Psi|_{y=H/2}$, the eigenvalues are found from the equation $d\Phi(p\Psi)/d\Psi|_{\Psi=\Psi_0}=0$.

We further consider the case $m(\Psi)=\Psi^{k-\mu}$, $0\le\Psi\le\Psi_0$; $Q(\Psi)=\Psi^{m-\delta}$, $\Psi_*\le\Psi\le\Psi_0$, where k and m are arbitrary integers, and $0\le\mu$, $\delta<1$. We denote $k+m-\mu-\delta\equiv n-\varepsilon$. The solution of Eq. (38) is then

$$\Phi(p_k\Psi)=\Psi^{\frac{1-a}{2}}Z_\nu\left(\frac{2}{M}p_k\Psi^{\frac{M}{2}}\right), \tag{41}$$

$$a=n-\varepsilon,\quad \nu=\frac{1-n+\varepsilon}{2-n+\varepsilon},\quad M=2-n+\varepsilon,\quad 1<a<2. \tag{42}$$

The norm of the transform is

$$N^2(p_k)=\int_0^{\Psi_0}\Phi^2(p_k\Psi)\,d\Psi=\frac{M}{4}\left(\frac{\gamma_k}{p_k}\right)^2\left[Z_\nu^2(\gamma_k)-Z_{\nu+1}(\gamma_k)\,Z_{\nu-1}(\gamma_k)\right],\quad \gamma_k=\frac{2}{M}p_k\Psi_0^{\frac{M}{2}}. \tag{43}$$

Here and above $Z\nu(\gamma)$ is a cylindrical function of order ν. (An error was committed in [7] in calculating the norm. The correct value of the factor in front of the square bracket in the denominator of expression (8) is $\frac{3}{8}\gamma_n^2\lambda_n^{-2}$.) Since,

$$\overline{C}(x,\ p_k)=\left\{\overline{C}(0,\ p_k)+\int_0^x\left[\frac{1}{\beta}\overline{W}(x,\ p_k)\,n(x)-K(p_k,\ x)\right]\exp\left[P(x,\ p_k)\right]dx\right\}\exp\left[-P(x,\ p_k)\right], \tag{44}$$

$$P(x,\ p_k)=\beta D_0p_k^2\int_0^x\frac{N(x)}{n(x)}dx-\frac{1}{\beta}\int_0^x Z(x)\,n(x)\,dx, \tag{45}$$

the required function is

$$C(x,\ \Psi)=\sum_{k=1}^{\infty}\Phi(p_k\Psi)\,\overline{C}(x,\ p_k)\,N^{-2}(p_k), \tag{46}$$

where the summation is performed over the increasing roots of the equation

$$\frac{1-a}{2}\Psi_0^{-\frac{1+a}{2}}Z_\nu(\gamma_k)+\frac{1}{2}\Psi_0^{\frac{-a+M-1}{2}}p_k\left[Z_{\nu-1}(\gamma_k)-Z_{\nu+1}(\gamma_k)\right]=0. \tag{47}$$

The transition to the (x, y) space is realized by replacing Ψ by the equivalent expression $\left[\frac{\beta}{n(x)}y\right]^{\frac{1}{1-k+\mu}}$.

It should be noted that the method suggested for solving Eq. (37) is also applicable for solving the equations of convective diffusion to spherical bodies in liquid or gas flow, such as drops, bubbles, and capillary-porous bodies of spherical shape. These equations have the form of relationship (37).

According to the general theory of eigenfunction expansion of the Sturm–Liouville problem [3], the coefficients of Eqs. (1), (26), (34), as well as the functions $\beta_1(t)$ and $\beta_2(t)$ do not have singularities, i.e., infinite discontinuities. Uniform convergence of the series (25) and (46), as well as the reality of the corresponding eigenvalues [3], follow from expansion theorems of this theory. The positiveness of these numbers follows from properties of Bessel functions and from the shape of the corresponding equations for finding them.

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STEADY-STATE TEMPERATURE DISTRIBUTION IN AN INHOMOGENEOUS MEDIUM WITH LOCAL INCLUSIONS

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UDC 536.24

We present a modification of the method of image regions [G. I. Marchuk, Methods of Numerical Mathematics, Springer-Verlag [1975)] to solve the boundary-value problem for the steady-state temperature distribution in an irregular multiply connected region.

We consider the boundary-value problem for the temperature distribution u(x) in the multiply connected region $G = \Pi \setminus \bigcup_{s=1}^{N} \omega_s$ (Fig. 1), where $\Pi = \{(x_1, x_2): 0 \le x_1 \le L, 0 \le x_2 \le l\}$, and ω_s is a region which corresponds to a local inclusion. At the boundary of the inclusion, the heat flux is zero:

$$\operatorname{div}[H(x)\operatorname{grad} u(x)] = -f(x),\ x = (x_1, x_2) \in G,$$
$$u|_\Gamma = 0,\ \left.\frac{\partial u}{\partial n}\right|_{\gamma_s} = 0\ (s = 1, 2, \ldots, N). \tag{1}$$

Here $H(x) > 0$ is the heat-conduction coefficient of the inhomogeneous medium; $f(x) > 0$, volume density of the heat sources; Γ, boundary of the rectangular region Π; γ_s, boundary of the local inclusion ω_s; and n, normal to the contour γ_s.

We shall present a method which makes it possible to find a rigorous solution of problem (1) for any shape and number of local inclusions ω_s. Together with (1) we shall formulate an auxiliary problem in the rectangular region Π:

$$\sum_{m=1}^{2} \frac{\partial}{\partial x_m}\left[\eta(x;\ \varepsilon)\frac{\partial v_\varepsilon}{\partial x_m}\right] = -F(x),\ x \in \Pi, \tag{2}$$

$$v_\varepsilon|_\Gamma = 0, \tag{3}$$

where $\eta(x; \varepsilon)$ and F(x) are piecewise-smooth functions which are defined as follows:

$$\eta(x;\ \varepsilon) = \begin{cases} H(x), & x \in G, \\ \varepsilon = \text{const} \geqslant 0, & x \in \Pi \setminus G, \end{cases} \quad F(x) = \begin{cases} f(x), & x \in G, \\ 0, & x \in \Pi \setminus G. \end{cases}$$

N. É. Bauman Higher Technical School, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 41, No. 1, pp. 158-163, July, 1981. Original article submitted May 12, 1980.

